Black holes in higher dimensions

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Abstract: Black holes in four dimensions have to be topologically spherical, dynamically stable and uniquely characterized by their conserved charges. In higher dimensions, black holes are less constrained. Black hole solutions have been found with non-spherical horizon topology, and the same conserved charges can describe different types of black objects. This work presents a study of known exact black hole solutions in higher dimensions: the generalization of the Schwarzschild and Kerr metric (Schwarzschild-Tangherlini and Myers-Perry rotating in a single plane) and a brief introduction to black p-branes.

I. INTRODUCTION

Some of the most promising approaches to unified theories of physics, such as string or braneworld theories, predict that spacetime has more than four dimensions. In these theories gravity is higher dimensional, so, naturally, they involve black holes in higher dimensions. This is one of the main drives to study this subject.

Black holes are fascinating objects. In 4 dimensions they exhibit remarkable features such as their complete characterization by mass and angular momentum, their spherical topology [1] and dynamical stability. It is of interest to study whether these properties hold for black holes in higher dimensions.

As we will see, General Relativity in higher dimensions has new and interesting aspects. Besides the straightforward generalization of the Schwarzschild solution to $d > 4$, there are new types of black objects with horizons that are not topological spheres. For example, we can find black strings and p-branes. These are black objects with infinitely extended horizons in p spatial directions, while compact in others. As shown by Gregory and Laflamme [2], black branes are dynamically unstable objects. The instability makes the horizon "ripple" and eventually pinch off, forming a naked singularity which violates cosmic censorship [3].

The Kerr black hole can also be generalized to higher dimensions, in the form of the Myers-Perry solution [4]. Adding extra spatial dimensions means that black holes can rotate in several independent planes and can have independent angular momenta. Whereas the angular momentum of a black hole in 4 dimensions is limited by the Kerr-bound, $J \leq GM$, in in $d \geq 6$ the Myers-Perry solutions have no upper bound: they can have arbitrarily large angular momenta. These are called ultra-spinning black holes, and they are also unstable [5].

In higher dimensions black objects are not uniquely characterized by their conserved charges. This was first shown by Emparan and Reall [6] with the example of a five-dimensional black hole with an event horizon of topology $S^1 \times S^2$, a black ring, whose rotation prevents collapsing. Within some ranges, the same values of mass and angular momentum can describe a rotating black hole with spherical topology and also two black rings. Therefore in high dimensions we find a violation of black hole uniqueness. As d increases, we expect to have a larger variety of black hole solutions. While there has been great progress in exploring them, it is believed that many more remain to be investigated [7].

Other interesting solutions have been found, such as a black saturn [8], a composite black object consisting of a spherical black hole surrounded by a black ring, superpositions of concentric black rings [9, 10], and compactifications of black branes called blackfolds [11], which are beyond the scope of this work.

II. ASYMPTOTICALLY FLAT SOLUTIONS

A. Conserved charges

A first, basic problem is how we can measure the mass and angular momentum of the solutions. Black holes are solutions to the Einstein equations that do not have any sources of mass; the matter stress tensor is zero. However, we can also identify the mass and angular momenta of isolated systems (such as black holes) from the asymptotic behavior of their gravitational field.

There are many different approaches to this problem (e.g., Hamiltonian analyses), but here we will follow one that is conceptually simple. The idea is that, at asymptotically large distances, the metric of the black hole should approach the metric of a solution to the linearized Einstein equations (in the weak field limit). Linear equations admit localized (distributional) sources. We can then consider a distributional stress energy tensor with given mass M and spins J_{ij} and, by solving the equations, we will obtain the weak-field geometry created at large distance by *any* localized object with mass M and spins J_{ij}. This will then allow us to extract the mass and spins from the asymptotic behavior of the metric components, with all the correct numerical proportionality

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coefficients.

We will solve Einstein's equations for this system following [7]. The metric takes the form of a small perturbation around flat Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

with $|h_{\mu\nu}| \ll 1$. We choose to work in transverse gauge

$$\nabla_\mu \bar{h}_{\mu\nu} = \nabla_\mu \left(h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \right) = 0, \tag{2}$$

where $\bar{h}_{\mu\nu}$ is the trace-reversed perturbation and h is the trace of $h_{\mu\nu}$.

With this choice, Einstein's equations are simply written to leading order as

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{3}$$

We will solve the equations for a point-like source of mass M and angular momentum with antisymmetric matrix J_{ij} at the origin $x^k = 0$ of flat space in Cartesian coordinates.

$$\begin{aligned} T_{tt} &= M \delta^{(d-1)}(x^k), & (4) \\ T_{ti} &= -\frac{1}{2} J_{ij} \nabla_j \delta^{(d-1)}(x^k). & (5) \end{aligned}$$

In spherical symmetry, the homogeneous equation $\Box \bar{h}_{tt} = 0$ takes the form

$$\frac{1}{r^{d-2}} \partial_r (r^{d-2} \partial_r \bar{h}_{tt}) = 0, \tag{6}$$

which is solved by

$$\bar{h}_{tt} = \frac{c}{r^{d-3}}, \tag{7}$$

where c is an integration constant which must be proportional to the mass M that sources the field. Our task now is to find the proportionality constant, when (7) is viewed as a solution with the distributional source (4).

For this purpose, we will integrate the two sides of the equation obtained from eq. (3) and (4),

$$\Box \bar{h}_{tt} = -16\pi G M \delta(x). \tag{8}$$

If we directly plugged (7) in the right hand side of this equation we would get 0. In order to catch the distributional source, we must work with this equation integrated over the volume on both sides. The right hand side is then integrated by parts, while the left hand side picks the delta. In this way we find

$$c = \frac{16\pi G M}{(d-3)\Omega_{d-2}}, \tag{9}$$

where Ω_{d-2} is the area of a unit $(d-2)$-sphere

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}, \tag{10}$$

as calculated in the appendix A.

We still need to find the complete solution for the metric perturbation $h_{\mu\nu}$, instead of its trace-reversed form. To find the solution for the perturbation,

$$\begin{aligned} \bar{h} &= \eta^{\mu\nu} \bar{h}_{\mu\nu} = \eta^{tt} \bar{h}_{tt} = -\bar{h}_{tt}; \\ \bar{h} &= -\frac{16\pi G M}{(d-3)\Omega_{d-2}} \frac{1}{r^{d-3}}. & (11) \end{aligned}$$

Since $h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2-d} \bar{h} \eta_{\mu\nu}$,

$$h_{tt} = \bar{h}_{tt} + \frac{1}{d-2} \bar{h} = \frac{d-3}{d-2} \bar{h}_{tt} = \frac{\mu}{r^{d-3}}, \tag{12}$$

where we have defined the mass parameter

$$\mu = \frac{16\pi G M}{(d-2)\, \Omega_{d-2}}, \tag{13}$$

which is convenient since it is the coefficient that appears directly in the (tt) metric component. This is one of our main results: if the gravitational field of an asymptotically flat solution decays at large distances like eq. (12), then we can extract its mass using eq. (13).

For completeness, we obtain that, in the transverse gauge in which we work, the spatial part takes the form

$$h_{ij} = \frac{1}{d-2} \bar{h}_{tt}\, \delta_{ij} = \frac{\mu}{(d-3)\, r^{d-3}}\, \delta_{ij}. \tag{14}$$

The previous calculation gives us the mass. Now we proceed to do a similar analysis for the angular momentum.

Since we are solving linear equations, the superposition principle allows us to study sources of mass and spin independently of each other. From (5)

$$\Box \bar{h}_{ti} = 8\pi G J_{ij} \nabla_j \delta^{(d-1)}(x). \tag{15}$$

This is solved by taking the Green's function and expanding with $r = |\mathbf{x}| \gg |\mathbf{y}|$

$$\begin{aligned} \bar{h}_{ti} &= \frac{16\pi G}{(d-3)\Omega_{d-2}} \int \mathrm{d}^{d-1} y \frac{T_{\mu\nu}(y^k)}{|\mathbf{x}-\mathbf{y}|^{d-3}} \\ &\approx -\frac{8\pi G}{\Omega_{d-2}} \frac{x^k}{r^{d-1}} \int \mathrm{d}^{d-1} y\, y^k J_{ij} \nabla_j \delta^{(d-1)}(y) \\ &= -\frac{8\pi G}{\Omega_{d-2}} \frac{x^k}{r^{d-1}} \int \mathrm{d}^{d-1} \delta_{kj} J_{ij} \delta^{(d-1)}(y) \\ &= -\frac{8\pi G}{\Omega_{d-2}} \frac{x^k J_{ik}}{r^{d-1}} = h_{ti}. & (16) \end{aligned}$$

In the last step we have used that this perturbation is traceless and therefore the barred and unbarred metric perturbations coincide.

Putting together the results of the two calculations, we find that the asymptotic metric of a solution with mass M and angular momentum J_{ij} is given by

$$h_{tt} = \frac{16\pi G}{(d-2)\,\Omega_{d-2}} \frac{M}{r^{d-3}}, \tag{17}$$

$$h_{ij} = \frac{16\pi G}{(d-2)(d-3)\,\Omega_{d-2}} \frac{M}{r^{d-3}} \delta_{ij}, \tag{18}$$

$$h_{ti} = -\frac{8\pi G}{\Omega_{d-2}} \frac{x^k J_{ik}}{r^{d-1}}. \tag{19}$$

Thus are defined the mass and angular momentum for any isolated system in its center of mass frame. The angular momentum J_{ij} can be simplified by a suitable coordinate rotation which allows us to write it in a block diagonal form

$$(J_{ij}) = \begin{pmatrix} 0 & J_{12} & 0 & 0 & \cdots \\ -J_{12} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & J_{13} & \cdots \\ 0 & 0 & -J_{13} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{20}$$

Each block corresponds to a plane of rotation. Following [7], angular momenta can be relabelled as $J_a = J_{2a-1,2a}$. There can be $N = \lfloor \frac{d-1}{2} \rfloor$ (integer part) independent angular momenta.

We can switch to polar coordinates for each of the planes,

$$(x_{2a-1}, x_{2a}) = (r_a \cos\phi_a, r_a \sin\phi_a). \tag{21}$$

The r_a are direction cosines for each independent rotation plane, and must satisfy $\sum_{a=1}^{N} r_a^2 = r^2$. Then

$$h_{t\phi_a} = -\frac{8\pi G J_a}{\Omega_{d-2}} \frac{r_a^2}{r^{d-1}}. \tag{22}$$

B. Schwarzschild-Tangherlini solution

Take the former solution and consider the static case. In polar coordinates, the metric takes the form

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\left(1 - \frac{\mu}{r^{d-3}}\right) \mathrm{d}t^2 \\ & + \left(1 + \frac{\mu}{(d-3)} \frac{1}{r^{d-3}}\right) \left(\mathrm{d}r^2 + r^2\, \mathrm{d}\Omega_{d-2}^2\right). \end{aligned} \tag{23}$$

This is very similar to the Schwarzschild metric except for the term that multiplies the angular part. The difference is simply due to a different choice of radial coordinate, since the conventional form of the Schwarzschild solution is not in transverse gauge. In order to relate them, we define ρ as the area radius,

$$\begin{aligned} \rho \;&=\; \left(1 + \frac{\mu}{(d-3)} \frac{1}{r^{d-3}}\right)^{1/2} r \\ &\approx\; \left(1 + \frac{\mu}{2(d-3)\, r^{d-3}}\right) r \\ &=\; \left(1 + \frac{8\pi G M}{(d-2)(d-3)\,\Omega_{d-2}\, r^{d-3}}\right) r. \end{aligned} \tag{24}$$

Plugging (24) in (23) and relabeling $\rho \to r$, the metric becomes

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-3}}\right) dt^2 + \left(1 - \frac{\mu}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2. \tag{25}$$

Since (23) was obtained as a solution to linear order in μ, in principle the steps from (23) to (25) would only give us a new form of the linear solution, and in particular the coefficient g_{rr} should be correct only to linear order in μ. However, it turns out that, in the form in (25), this is an exact solution to the Einstein equations in d dimensions.

This solution was found by Tangherlini [12] in 1963 and it is a generalization to higher dimensions of the Schwarzschild solution. For $d = 4$, $\mu = 2GM$, we recover the familiar solution.

For $\mu > 0$, the surface $r^{d-3} = \mu$ is an event horizon. It is possible to change to Kruskal-type coordinates that are regular on the event horizon and find the maximal analytic extension. The corresponding Penrose diagram takes the exact form as its 4 dimensional counterpart, but now each point represents a $(d-2)$-sphere.

It does not present new features respect to the 4-dimensional case. In fact, Birkhoff's theorem can be extended to higher dimensions [13], yielding that Schwarzschild-Tangherlini is the most general solution for an asymptotically flat and hyperspherically symmetric geometry.

There is also a uniqueness theorem [14] that yields that the Schwarzschild-Tangherlini metric is the only solution of the vacuum Einstein equations in higher dimensions for an asymptotically flat and static geometry.

Therefore, in an extension of the four-dimensional result, a higher-dimensional static, neutral black hole is fully characterized by its mass.

C. Myers-Perry black holes

The Kerr metric can be also generalized. The d dimensional solution describing a black hole rotating in all possible independent rotation planes was found by Myers and Perry [4] in 1986. We will only discuss the case of a black hole rotating in a single plane. The metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = & - \mathrm{d}t^2 + \frac{\mu}{r^{d-5}\Sigma} (\mathrm{d}t - a \sin^2\theta\, \mathrm{d}\phi)^2 \\ & + \frac{\Sigma}{\Delta} \mathrm{d}r^2 + \Sigma\, \mathrm{d}\theta^2 + (r^2 + a^2) \sin^2\theta\, \mathrm{d}\phi^2 \\ & + r^2 \cos^2\theta\, \mathrm{d}\Omega_{d-4}^2, \end{aligned} \tag{26}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - \frac{\mu}{r^{d-5}}. \tag{27}$$

Our previous results now allow us to obtain the mass and angular momentum of this solution. For this purpose, we simply need to expand the metric components g_{tt} and $g_{t\phi}$ at large distances, $r \gg 1$.

$$h_{tt} = \frac{\mu}{r^{d-5}\Sigma}, \tag{28}$$

$$h_{t\phi_a} = -\frac{2a\sin^2\theta\mu}{r^{d-5}\Sigma}. \tag{29}$$

Comparing this to (17) and (22) in the asymptotic field, yields

$$M = \frac{(d-2)\,\Omega_{d-2}}{16\pi G}\mu, \tag{30}$$

$$J_i = \frac{2}{d-2}Ma_i. \tag{31}$$

This result tells us that the parameter a in the Myers-Perry solution is the angular momentum per unit mass, up to a d-dependent factor. Setting $a_i = 0$ we recover the Schwarzschild-Tangherlini solution.

In many respects this metric is similar to the Kerr solution. The last term corresponds to the line element on a $(d-4)$-sphere, it represents the additional spatial dimensions and vanishes for the case $d = 4$. The other difference we notice is that the radial fall-off of the gravitational potential depends on the number of dimensions.

Even if this might seem innocuous, it is actually the source of a main difference between the Kerr black holes and the Myers-Perry black holes. The reason is the contrast with the behavior of the centrifugal repulsion, which does not depend on the number of dimensions, since rotation happens on a plane. The competition between gravitational attraction and centrifugal repulsion will then have a strong dependence on the dimensionality, which we can heuristically see with

$$\frac{\Delta}{r^2} - 1 = -\frac{\mu}{r^{d-3}} + \frac{a^2}{r^2}. \tag{32}$$

The first term on the right is the attractive Newtonian potential in d spacetime dimensions. The second is the repulsive centrifugal potential, which as we can see does not depend on d. Therefore the competition between the two terms will change with d.

The outer event horizon is located at the r_0 that makes $g^{rr} = 0$, that is, $\Delta(r) = 0$.

$$r_0^2 + a^2 - \frac{\mu}{r_0^{d-5}} = 0. \tag{33}$$

Since Δ depends on the number of dimensions, this means that the features of the event horizons will depend on the dimension.

Setting $d = 4$ we get the familiar second order equation

$$r_0^2 - \mu r_0 + a^2 = 0, \tag{34}$$

$$r_0 = \frac{\mu \pm \sqrt{\mu^2 - 4a^2}}{2} = GM \pm \sqrt{G^2M^2 - a^2}. \tag{35}$$

Which yields that we will have a regular event horizon for $a < GM$. For the extremal case, $a = GM$, the Kerr-bound, there is a single degenerate horizon, and the case with $a > GM$ features a naked singularity. Similarly, in $d = 5$ we get $r_0 = \pm\sqrt{\mu - a^2}$, yielding that the solution only exists up to $a^2 = \mu$.

This is not the case for $d \geq 6$. The function $\Delta(r)$ is positive for large r and negative for small r, so (Bolzano's theorem) there will always be a single positive root of $\Delta(r) = 0$, independently of the value of a. Therefore a has no bound, it can take any arbitrarily large value. Myers-Perry black holes with large a are known as *ultra-spinning* black holes. As mentioned in the introduction, these are unstable [5, 15], as they spread out in the plane of rotation and become a black membrane with horizon geometry $\mathbb{R}^2 \times S^{d-4}$ which experiment Gregory-Laflamme-like instabilities.

III. BLACK P-BRANES

The easiest way to construct a higher dimensional black object is to add a flat spatial direction to a known black hole vacuum solution of Einstein's equations. For example, take Schwarzschild solution in $d = 4$ and add an extra spatial dimension,

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2 + \mathrm{d}z^2. \tag{36}$$

It is also a Ricci flat manifold, since it is the direct product of two Ricci-flat manifolds. This is called a black string and it has an event horizon with topology $S^2 \times \mathbb{R}$. This simple construction shows that horizons need not to be spherical. Furthermore, if a piece of black string is bent into a circular shape, we would have a *black ring*, with topology $S^2 \times S^1$, which should rotate in order to prevent the collapse.

In general, we can add p flat spatial directions to any vacuum black hole solution $\mathcal{B}$ in d dimensions with horizon topology $\mathcal{H}$, which will have an horizon with topology $\mathcal{H} \times \mathbb{R}^p$: these are black p-branes. They do not have a counterpart in $d = 4$.

We can find other topologies, for example, if we identify $x^i \sim x^i + L_i$, getting a topology $\mathcal{H} \times \mathbb{T}^p$.

There are, however, some restrictions on the topology of the event horizons [16], which we will not review.

IV. CONCLUSIONS

Black holes in higher dimensions are a thrilling new chapter in general relativity. They can be radically different compared to their four-dimensional counterparts:

they present new horizon topologies, they have hair (they are not uniquely characterized by their conserved charges), and some of them are unstable. In this work we have barely had a glimpse of the rich variety of solutions that can be found in higher dimensions.

In particular, we have seen how we can define conserved charges to asymptotically flat solutions. We have reviewed the generalization of the Schwarzschild solution to the higher dimensional case, we have presented the solution for a Myers-Perry black hole rotating in a single plane and, using our previous results, we have extracted their physical mass and angular momentum from the asymptotic behavior of the solutions. The surprising fact that for $d \geq 6$ the angular momentum can be arbitrarily large has been emphasized. Finally, we have briefly and heuristically shown how other types of black objects can be constructed.

Appendix A: Area of a unit n-sphere

We begin by writing down the metric of a 2-sphere and a 3-sphere hoping to recognize a pattern.

$$\begin{aligned} n=2 &\;\rightarrow\; \mathrm{d}\Omega_2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\theta_2^2 \\ n=3 &\;\rightarrow\; \mathrm{d}\Omega_3 = \mathrm{d}\theta_1^2 + \sin^2\theta_1(\mathrm{d}\theta_2^2 + \sin^2\theta_2\;\mathrm{d}\theta_3^2) \\ &\;\vdots \\ n &\;\rightarrow\; \mathrm{d}\Omega_n = \mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\Omega_{n-1} \end{aligned} \tag{A1}$$

In order to find Ω_n we just have to integrate the former recurrence relation,

$$\begin{aligned} \Omega_n &= \int_0^\pi \mathrm{d}\theta_1 \sin^{n-1}\theta_1\,\Omega_{n-1} = \Omega_{n-1}\int_0^\pi \mathrm{d}\theta_1 \sin^{n-1}(\theta_1) \\ &= 2\;\Omega_{n-1}\int_0^{\pi/2} \mathrm{d}\theta_1 \sin^{2\left(\frac{n}{2}\right)-1}(\theta_1)\cos^{2\left(\frac{1}{2}\right)-1}(\theta_1) \\ &= 2\;\Omega_{n-1}\mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right) = 2\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\Omega_{n-1} \\ &= 2\sqrt{\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\Omega_{n-1} = 2\pi\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\Omega_{n-2} \\ &= \cdots = 2\pi^{n/2}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \end{aligned} \tag{A2}$$

An alternative derivation with gamma function as starting point can be found in [17].

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